

# Geometry of the Motion of Ideal Fluids and Rigid Bodies

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## Abstract

Arnold pointed out that the Euler equation of incompressible ideal hydrodynamics describes geodesics on the group of volume-preserving diffeomorphisms. A simple analogue is the Euler equation for a rigid body, which is the geodesic equation on the rotation group with respect to a metric determined by the moment of inertia. The metric on the group is left-invariant but not right-invariant. We will reduce the geometry of such groups (using techniques popularized by Milnor) to algebra on their tangent space. In particular, the curvature can be expressed as a biquadratic form on the Lie algebra. Arnold's result that motion of incompressible fluids has instabilities (due to the sectional curvature being negative) can be recovered more simply. Surprisingly, such an instability arises in rigid body mechanics as well: the metric on  $SO(3)$  corresponding to the moment of inertia of a thin cylinder (coin) has negative sectional curvature in one tangent plane.

Both ideal fluids and rigid bodies can be thought of as hamiltonian systems with a quadratic hamiltonian, but whose Poisson brackets are those of a non-nilpotent Lie algebra. We will also describe a different point of view towards three dimensional incompressible flow in terms of the Clebsch parametrization. In this picture, the Poisson brackets are represented canonically. The hamiltonian is represented by a quartic function.

This is meant mainly as an expository article, aimed at a mathematical audience familiar with physics. Based on Lectures at the Chennai Mathematical Institute and the University of Connecticut.

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# 1 Introduction

Unlike in mathematics, the problems of physics tend not to be very old. Experimental advances constantly invalidate old ideas or introduce completely new ones. However, the problem of understanding a non-integrable dynamical system ('chaos') is as old as physics itself and is still largely unsolved. A particularly virulent example is the phenomenon of turbulence in fluid mechanics: when velocity exceeds a critical value, the flow suddenly becomes irregular and unpredictable except by very fine numerical methods. A theoretical understanding of this phenomenon, perhaps along the lines of Wilson's theory of second order phase transitions, remains a great challenge.

A separate question is whether the partial differential equations of hydrodynamics (Navier-Stokes or Euler) have unique solutions and how regular they are. This has been recognized as a mathematical challenge worthy of the best analysts. Even though the two problems in physics and mathematics are different, one hopes that ideas from one will cross-fertilize the other.

We begin by reviewing the basic equations of the subject. For simplicity and brevity, it is hard to beat the classic text by Landau and Lifshitz [1]. For the geometrical formulation, the references are the books by Arnold and Khesin [?] and Kambe [4]. The expository article by Milnor [8] shows how the geometry of a group can be reduced to algebraic questions on its tangent space. This will be important for us because it allows us to avoid defining an infinite dimensional manifold. We can reduce everything to vector spaces and linear operators on them: much simpler technically, while the geometry of manifolds continues to provide powerful intuition in the infinite dimensional case.

## 1.1 Ideal Fluids

We will consider only ideal non-relativistic fluids; that is, a fluids whose velocities everywhere are small compared to the velocity of light and in which losses due to friction (viscosity) are small enough to be ignored.

### 1.1.1 The Two Time Derivatives

There are two ways of thinking about the time dependence of any physical quantity in a fluid: at a fixed location in space or, along the flow of the fluid. The first is the partial derivative  $\frac{\partial}{\partial t}$  and the second is the total (or material) derivative  $\frac{d}{dt}$ . They are related by

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \mathcal{L}_v$$

where  $v$  is the velocity of the fluid and  $\mathcal{L}_v$  is the Lie derivative. On a scalar field it is just

$$\frac{d\phi}{dt} = \frac{\partial\phi}{\partial t} + v^i \partial_i \phi$$

We will also be interested in the case of a density. A density on a manifold is simply a differential form of the highest possible rank ( i.e., equal to the dimension of the manifold),  $\rho dx^1 \wedge \cdots dx^n$ . Although it has one independent component like a scalar, the Lie derivative of a density is different from that of a scalar:

$$\mathcal{L}_v \rho = \partial_i [\rho v^i].$$

The divergence of a vector field can be defined as

$$\text{div } v = \frac{1}{\rho} \mathcal{L}_v \rho = \frac{1}{\rho} \partial_i [\rho v^i].$$

The time derivative of the velocity field itself requires a different idea. Since  $\mathcal{L}_v v = [v, v] = 0$ , the Lie derivative does not capture its variation due to the motion of the fluid. Using a Riemannian metric  $g$  of the manifold in which the fluid is moving ( more precisely, using its Levi-Civita connection  $\nabla$ ) we can find the acceleration of a fluid element as  $\frac{dv}{dt} = \frac{\partial v}{\partial t} + \nabla_v v$

$$\frac{dv^i}{dt} = \frac{\partial v^i}{\partial t} + v^j \nabla_j v^i$$

where, as usual,

$$\nabla_i v^j = \partial_i v^j + \Gamma_{ik}^j v^k, \quad \Gamma_{ik}^j = \frac{1}{2} g^{jl} [\partial_i g_{lk} + \partial_k g_{il} - \partial_l g_{ik}].$$

### 1.1.2 Conservation of Mass

The first law of motion of the fluid is just the conservation of mass:

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x^i} (\rho v^i) = 0 \tag{1}$$

We can write this also as

$$\frac{d\rho}{dt} = \frac{\partial \rho}{\partial t} + \mathcal{L}_v \rho = 0$$

in terms of the material derivative. It is clear that this equation does not make use of the Riemannian metric of the manifold  $M$  in which the fluid is moving.

### 1.1.3 Conservation of Momentum

For a fluid without external forces, Newton's second law gives

$$\rho \frac{dv_i}{dt} = -\nabla_i p$$

where  $p$  is the pressure and  $\frac{d}{dt}$  is the derivative taken in the co-moving reference frame of the fluid. It includes the explicit time derivative as well as the change due to the motion of the fluid element:

$$\frac{\partial v_i}{\partial t} + v^k \nabla_k v_i = -\frac{1}{\rho} \nabla_i p. \quad (2)$$

It can be re-expressed as the conservation of momentum density:

$$\frac{\partial(\rho v_i)}{\partial t} + \nabla_k T_i^k = 0 \quad (3)$$

where the stress tensor density is

$$T_i^k = p \delta_i^k + \rho v^k v_i.$$

This equation does use the metric of the underlying space, which is often taken to be Euclidean in physical applications. But there are physically interesting cases of fluid motion on a curved geometry as well: the ocean or the atmosphere of a planet can often be thought of as a fluid on the surface of a sphere.

#### 1.1.4 Equation of State

So far we have one scalar equation and one vector equation for the unknown quantities  $\rho, p, v^i$ . We need one more scalar equation to have enough information to determine them. This is given by an equation of state: a relation between pressure and density. A model that works well in many situations is the law  $p = A\rho^\gamma$ , (polytrope) for some constants  $A, \gamma$  characteristic of the fluid. For the atmosphere in the adiabatic approximation,  $\gamma \approx 1.4$ .

#### 1.1.5 The Wave Equation of Sound

Before studying any non-linear equation in depth, we must understand its linear approximation. If the gradient of the velocity is small and the departure  $\rho_1$  of the density from some average value  $\rho_0$  is small, the above equations linearize to

$$\frac{\partial \rho_1}{\partial t} + \partial_i [\rho_0 v^i] = 0$$

$$\frac{\partial(\rho_0 v^i)}{\partial t} + \kappa \nabla^i \rho_1 = 0.$$

where  $\kappa = \left[ \frac{\partial p}{\partial \rho} \right]_{\rho=\rho_0}$ . By differentiating the first equation w.r.t. time and putting in the second we get

$$\frac{\partial^2 \rho_1}{\partial t^2} - \kappa \nabla_i \nabla^i \rho_1 = 0$$

which is the equation for some wave propagating with velocity  $c = \sqrt{\kappa}$ . These are sound waves. Thus sound is the infinitesimal manifestation of fluid flow.

### 1.1.6 Isentropic Flows

If there is a function  $w$  ('enthalpy') such that

$$\partial_i w = \frac{1}{\rho} \partial_i p$$

the flow is said to be isentropic. It basically means that no heat is lost or gained by the system. Then the second equation of motion can be written as

$$\frac{\partial v_i}{\partial t} + v^k \nabla_k v_i + \nabla^i w = 0. \quad (4)$$

We still need an equation of state giving  $w$  as a function of  $\rho$ .

## 1.2 Incompressible Fluids

If the time dependence of the density is small enough to be ignored,

$$\frac{\partial \rho}{\partial t} = 0$$

we say that it is *incompressible*. Then compressibility  $\kappa$  tends to infinity: the speed of sound is infinite. More precisely, the speed of the fluid flow is small compared to the speed of sound. In this case we get the equations of Euler:

$$\partial_k [\rho v^k] = 0 \quad (5)$$

$$\frac{\partial v^i}{\partial t} + v^j \nabla_j v^i + \frac{\nabla^i p}{\rho} = 0.$$

The density and the metric of the manifold containing the fluid should be given;  $p$  is a Lagrange multiplier enforcing the condition of incompressibility 5.

### Incompressible does not mean constant density

It is often stated incorrectly that an incompressible fluid must have constant density. Incompressibility only means that the density is given as a function of space and is independent of time: it is not a dynamical variable. The atmosphere of the Earth, for example, has density decreasing with height; yet the atmospheric flows (winds) have velocities much lower than the speed of sound. The atmosphere is an incompressible fluid.

#### 1.2.1 Boundary Conditions

We have not said much about the underlying manifold  $M$  on which the fluid flows. In most cases of interest it is a domain of Euclidean space. But it is useful to consider the more general case of a Riemannian manifold, as the mathematical concepts are more clear then. Even in physics, occasionally one is interested in the flow of a fluid on a curved manifold such as the sphere (ocean

currents). In general the manifold  $M$  will have a boundary. For an ideal fluid, it is sufficient that the velocity field be tangential to the boundary.

In the real world, dissipation (viscosity) cannot be ignored near the boundary, even if it is small away from the boundary. Thus, the physically correct boundary condition is that the velocity field must vanish at the boundary, not just its tangential component. We will assume this stronger condition in what follows.

Thus, all vector fields will vanish at the boundary. All diffeomorphisms reduce to the identity at the boundary.

### 1.2.2 Two Dimensional Incompressible Fluids

A particularly interesting example is the case of incompressible fluids in two dimensions. Geophysical applications include ocean currents and atmospheric flows which are approximately two dimensional: the atmosphere and the oceans have a depth small compared to the diameter of the Earth. As an example we consider the case of flow in the plane and with  $\rho = 1$ : a part of the sphere that is small enough that the curvature can be ignored. We will return to a more general theory later.

We can eliminate  $w$  by taking the curl of the above equation. In two dimensions, the curl of velocity ('vorticity') is a scalar

$$\omega = \partial_1 v_2 - \partial_2 v_1.$$

Since

$$\partial_1(v_j \partial_j v_2) - \partial_2(v_j \partial_j v_1) = \partial_1 v_j \partial_j v_2 - \partial_2 v_j \partial_j v_1 + v_j \partial_j \omega \quad (6)$$

$$= \partial_1 v_1 \partial_1 v_2 + \partial_1 v_2 \partial_2 v_2 - \partial_2 v_1 \partial_1 v_1 - \partial_2 v_2 \partial_2 v_1 + v_j \partial_j \omega \quad (8)$$

$$= \omega(\partial_j v_j) + v_j \partial_j \omega \quad (9)$$

$$= v_j \partial_j \omega$$

Moreover, every incompressible vector field is of the form

$$v_1 = \partial_2 \chi, \quad v_2 = -\partial_1 \chi$$

for some 'stream function'  $\chi$ . Vorticity is then

$$\omega = -\partial_1^2 \chi - \partial_2^2 \chi \equiv \Delta \chi.$$

where  $\Delta$  is a positive laplacian. Given appropriate boundary conditions, this is an invertible operator. Thus we can regard the vorticity as the dynamical variable and the velocity potential as derived from it by solving the above elliptic differential equation.

Also,

$$v_j \partial_j \omega = \partial_2 \chi \partial_1 \omega - \partial_1 \chi \partial_2 \omega = \{\chi, \omega\}$$

which is the Poisson bracket on a two dimensional phase space. It is anti-symmetric and satisfies the Jacobi identity. Thus, two dimensional incompressible flow reduces to the pair of equations

$$\frac{\partial \omega}{\partial t} + \{\chi, \omega\} = 0, \quad \omega = \Delta \chi.$$

The Green's function of the laplacian (with appropriate boundary conditions), can be thought of a linear operator which solves the Poisson equation  $\chi = K\omega$ . Then the equation of motion of an incompressible fluid in two dimensions becomes the non-linear integro-differential equation

$$\frac{\partial \omega}{\partial t} + \{K\omega, \omega\} = 0. \quad (10)$$

## 2 The Rigid Body

It is useful to start with an example of a mechanical system that looks like the opposite extreme from a fluid: a rigid body. We will see that there are many similarities in the basic mathematical formulation, although fluid mechanics is much more complicated. It is interesting that the basic equations of both extremes are due to Euler.

### 2.1 Euler Equations

Recall the Euler equations of a rigid body on which no external forces are acting are an expression of the conservation of angular momentum  $\mathbf{L}$ . In the non-inertial reference attached to the body itself, this takes the form

$$\frac{d\mathbf{L}}{dt} + \boldsymbol{\Omega} \times \mathbf{L} = 0$$

where  $\boldsymbol{\Omega}$  is the angular velocity. These quantities are related by a positive symmetric tensor (linear operator),  $I$  the moment of inertia:

$$\mathbf{L} = I\boldsymbol{\Omega}.$$

The moment of inertia  $I$  is defined in terms of the density of the rigid body as follows:

$$I_{ij} = \delta_{ij} M_{kk} - M_{ij}, \quad M_{ij} = \int \rho(x) x_i x_j dx,$$

It is obvious that the matrix  $M \geq 0$ . Suppose its eigenvalues are  $M_1, M_2, M_3$ , all positive numbers. Then  $M_{kk} = M_1 + M_2 + M_3$ . The matrices  $M$  and  $I$  are diagonal in the same basis, and the eigenvalues of  $I_{ij}$  are

$$I_1 = M_2 + M_3, \quad I_2 = M_1 + M_3, \quad I_3 = M_1 + M_2.$$

Thus  $I \geq 0$  as well. The basis in which  $I$  is diagonal forms the principal axes of the body and  $I_1, I_2, I_3$  are called the principal moments of inertia.

If we compare with the Euler equations in two dimensions, we see that the vorticity is analogous to angular momentum and the velocity potential analogous

to angular velocity. Also, the elliptic differential operator  $\Delta$  above is analogous to the moment of inertia. The cross product of vectors (which is anti-symmetric and satisfies the Jacobi identity) corresponds to the Poisson bracket of functions.

$$L \longleftrightarrow \omega, \quad \Omega \longleftrightarrow \chi, \quad I \longleftrightarrow \Delta, \times \longleftrightarrow \{, \}.$$

Of course, the angular momentum has only three independent components while vorticity belongs to an infinite dimensional space. So fluid mechanics is much more complicated.

There is a co-ordinate system, which moves together with the body, in which  $I_{ij}$  is diagonal:

$$L_1 = I_1 \Omega_1, \quad L_2 = I_2 \Omega_2, \quad L_3 = I_3 \Omega_3.$$

In terms of these the Euler equations become

$$\frac{d\Omega_1}{dt} + \frac{I_2 - I_3}{I_1} \Omega_2 \Omega_3 = 0, \quad \frac{d\Omega_2}{dt} + \frac{I_3 - I_1}{I_2} \Omega_3 \Omega_1 = 0, \quad \frac{d\Omega_3}{dt} + \frac{I_1 - I_2}{I_3} \Omega_1 \Omega_2 = 0 \quad (11)$$

They are solved in terms of the three Jacobi elliptic functions .

### 2.1.1 Hamiltonian Formalism

A rigid body moves so that the distance between any two points on it remains fixed. Thus its configuration space is the isometry group of  $R^3$ . More precisely the connected component of the isometry group, which consists of rotations and translations. The translations are uninteresting as they simply describe a straight-line on which the center of mass moves. Thus, we can think of the group of rotations  $SO(3)$  as the configuration space of a rigid body.

The components of angular momentum satisfy the Poisson bracket relations

$$\{L_1, L_2\} = L_3, \quad \{L_2, L_3\} = L_1, \quad \{L_3, L_1\} = L_2.$$

Just as the kinetic energy due to translational motion of a particle is  $\frac{\mathbf{P}^2}{2m}$ , the rotational kinetic energy of a rigid body is given by

$$H = \frac{L_1^2}{2I_1} + \frac{L_2^2}{2I_2} + \frac{L_3^2}{2I_3} \quad (12)$$

in the basis where  $I_{ij}$  is diagonal. Indeed, we can check that Euler equations above are implied by this hamiltonian and the above Poisson brackets:

$$\frac{dL_i}{dt} = \{H, L_i\}.$$



### 2.1.2 The Elliptic Curve

The Poisson algebra of angular momentum has a non-trivial center. That is, there is a polynomial in the generators that commutes with (has zero Poisson brackets with) all the generators:

$$L^2 = L_1^2 + L_2^2 + L_3^2. \quad (13)$$

It is the square of the magnitude of angular momentum. In particular, it commutes with the hamiltonian and hence is a conserved quantity:

$$\frac{dL^2}{dt} = 0.$$

Of course, the hamiltonian is itself a conserved quantity.

Thus the solution to the Euler equations describes parametrically the curve which is the intersection of the sphere (13) with the ellipsoid (12). This intersection can be either a union of disjoint circles immersed in  $R^3$  or a single connected closed curve, depending on the values of  $H$  and  $L$ .

So the solution of Euler's equation has a purely algebraic description. When solving algebraic equations, it is useful to continue to complex values, even if the physical values are real as in our case. The intersection curve is then a complex curve, a manifold of two real dimension. A moment's thought will convince you that this manifold must be a torus: it is compact, and the time evolution defines an everywhere non-zero vector field on it. (Only connectedness needs a proof which we skip.) Thus was born the theory of elliptic curves.

The theory of elliptic curves have been honed into a fine marble sculpture in the garden of mathematics. But some of the life has been lost in this process: the physics seems to be lost.

### 2.1.3 Geodesics on $SO(3)$

Euler equations have a natural geometric interpretation as geodesics on the rotation group, with respect to a metric determined by the moment of inertia. This metric is, in interesting cases, not bi-invariant. Instead, it is only invariant under the left action. Thus we can visualize the universal cover of the rotation group as a three dimensional ellipsoid.

To see where this metric comes from, remember that the Lie algebra of a group can be thought of as the space of left-invariant vector fields. A positive quadratic form on the Lie algebra defines a left invariant metric on the group. Euler's equations describe how the tangent vectors to the geodesics evolve, with 'time' having the meaning of arc length.

This point of view will be very useful for us in understanding hydrodynamics. So we will describe in more detail the geometry of left-invariant metrics in a later section.

### 3 Hamiltonian Systems From Lie Algebras

We will see that the Euler equation of both ideal fluid mechanics and the rigid body are special cases of a class of dynamical systems obtained from a real Lie algebra  $\mathcal{G}$ . We digress a bit to describe this class of systems.

The set of observables of a classical mechanical system is a *Poisson algebra*. That is, a commutative algebra on which is defined in addition a bilinear (Poisson bracket) such that

1.  $\{f_1, f_2\} = -\{f_2, f_1\}$  anti-symmetry
2.  $\{\{f_1, f_2\}, f_3\} + \{\{f_2, f_3\}, f_1\} + \{\{f_3, f_1\}, f_2\} = 0$  Jacobi identity
3.  $\{f_1, f_2 f_3\} = \{f_1, f_2\} f_3 + f_2 \{f_1, f_3\}$  Leibnitz identity

The algebra of functions  $f : \mathcal{G}^* \rightarrow R$  on the dual of a Lie algebra is an example of a Poisson algebra. The exterior derivative of such a function can be thought of as valued in the dual of  $\mathcal{G}^*$ , which can be identified with  $\mathcal{G}$ . Thus it makes sense to take the Lie bracket of a pair of such exterior derivatives  $[df_1(a), df_2(a)]$  evaluated at some point  $a \in \mathcal{G}^*$ . A contraction with  $a$  itself gives a number. So we define the Poisson bracket to be

$$\{f_1, f_2\}(a) = -i_a([df_1(a), df_2(a)]).$$

The required properties follow from those of a Lie algebra and the exterior derivative.

If we choose a particular function  $H$  as the hamiltonian, we get a dynamical system for which the time evolution of any observable is given by

$$\frac{df}{dt} = \{H, f\}.$$

Now, every Lie algebra element  $\omega \in \mathcal{G}$  defines a linear function on its dual. In this case, we get a simpler form of this equation:

$$\frac{d\omega}{dt} + [dH, \omega] = 0.$$

#### 3.1 Metric Lie Algebra

A Metric Lie Algebra is a Lie algebra along with an inner product (usually not invariant). Now, an inner product is a symmetric tensor on  $\mathcal{G}$ , and hence can be thought of as a quadratic function on  $\mathcal{G}^*$ . If we choose this function as the hamiltonian,  $dh : \mathcal{G}^* \rightarrow \mathcal{G}$  will be a linear function; which is another way of thinking of an inner product.

If our Lie algebra  $\mathcal{G}$  admits an invariant inner product  $\langle, \rangle$  (not necessarily  $H$ ), we can write this equation a bit more explicitly. Then we can use the invariant inner product to identify  $\mathcal{G}^*$  and  $\mathcal{G}$  and  $dH$  becomes just a linear operator  $K : \mathcal{G}^* \rightarrow \mathcal{G}$  :

$$H(\omega) = \frac{1}{2} \langle \omega, K\omega \rangle .$$

The equations of motion are then

$$\frac{d\omega}{dt} + [K\omega, \omega] = 0.$$

An example of this is the rigid body, where the Lie algebra is the cross product, the invariant inner product is the dot product and  $K$  is the inverse of the moment of inertia, yielding Euler's equations .

When the Lie algebra admits no invariant inner product it is not as easy to write an explicit form for the equations. It is then useful to describe it in a basis  $L_i$  of the Lie algebra. The Poisson bracket is determined completely by its effect on the generators:

$$\{L_i, L_j\} = c_{ij}^k L_k$$

where  $c_{ij}^k$  are the structure constants of the Lie algebra in this basis. Thus for example,

$$\{L_i, L_j L_m\} = c_{ij}^k L_k L_m + c_{im}^k L_j L_k$$

We are thinking of the basis elements of  $\mathcal{G}$  as generators of the algebra of polynomials on  $\mathcal{G}^*$ . This is a formal translation of the point of view of most physicists.

A particularly simple polynomial is given by an inner product on the Lie algebra. Expressed in the above basis,

$$H = \frac{1}{2} h^{ij} L_i L_j$$

where  $h^{ij} = h^{ji}$ . If we use this as a hamiltonian, the equations of time evolution

$$\frac{df}{dt} = \{H, f\} .$$

can be written as

$$\frac{dL_k}{dt} = h^{ij} c_{ik}^m L_m L_j .$$

If the inner product  $h$  were invariant, it would satisfy the identity

$$h^{ij} c_{ik}^m + i \leftrightarrow j = 0$$

and there would be no time evolution.

### 3.2 The Metric Lie Algebra of a Two Dimensional Incompressible Fluid

The set of functions on the plane form a Lie algebra under the Poisson bracket

$$\{f, g\} = \partial_2 f \partial_1 g - \partial_1 f \partial_2 g.$$

There an invariant inner product in this Lie algebra

$$\langle f, g \rangle = \int f g dx.$$

If choose as hamiltonian the positive quadratic form

$$H(\omega) = \frac{1}{2} \langle \omega, G\omega \rangle$$

where  $G$  is the Green's function of the Laplace operator  $\Delta$ , the equations of motion obtained are exactly those of an incompressible fluid in two dimensions.<sup>1</sup>

Thus the inner product that defines an incompressible fluid is

$$\int f(x) G(x, y) g(y) dx dy$$

where

$$(\partial_1^2 + \partial_2^2) G(x, y) = -\delta(x, y)$$

with appropriate boundary conditions.

### 3.3 Dimensions Greater Than Two

Even if the dimension is greater than two, the hamiltonian of incompressible fluid is still a quadratic function on the Metric Lie Algebra of incompressible vector fields. However there is no longer any obvious invariant inner product, so we have to contend with describing the equations a bit indirectly.

The condition of incompressibility

$$\partial_k (\rho v^k) = 0 \tag{14}$$

is equivalent to

$$\mathcal{L}_v \rho = 0.$$

It follows that the commutator of two incompressible vector fields is also incompressible: the set of solutions  $\mathcal{V}_\rho$  of 14 is a Lie algebra with the same Lie bracket as before. A linear function on  $\mathcal{V}_\rho$  is of the form  $\int \rho v^i a_i dx$  for some 1-form  $a$ . But, a gauge transformation

$$a \mapsto a + d\Lambda$$

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<sup>1</sup>We must assume appropriate boundary conditions on  $\omega$  so that  $H(\omega)$  exists.

leaves this function unchanged. Thus the dual of  $\mathcal{V}_\rho$  is the space of 1-forms modulo exact 1-forms:

$$\mathcal{V}_\rho^* \equiv \Lambda^1 / d\Lambda^0.$$

The kinetic energy of the fluid is a simple quadratic form on the Lie algebra of incompressible vector fields:  $\frac{1}{2} \int \rho v^i v^j g_{ij} dx$ . The dual variable dual to  $v^i$  can be thought of as just

$$a_i = [g_{ij} v^j]$$

the square brackets being there to remind us that we must take the equivalence classes under gauge transformations. Thus the Hamiltonian can only depend on the vorticity

$$\omega = da, \quad \omega_{ij} = \partial_i a_j - \partial_j a_i.$$

But it must be a zeroth order operator in terms of velocity.

$$H = \frac{1}{2}(\omega, G\omega)$$

where  $G$  is the Green's function of the elliptic system

$$\omega = da, \quad \partial_i (\rho g^{ij} a_j) = 0.$$

The Euler equations in vorticity form are

$$\frac{\partial \omega}{\partial t} + \mathcal{L}_v \omega = 0$$

where we are to regard  $v$  as the unique incompressible vector field determined by the vorticity by

$$\partial_i [\rho v^i] = 0, \quad \partial_i [g_{jk} v^k] - \partial_j [g_{ik} v^k] = \omega_{ij}$$

with appropriate boundary conditions.

## 4 Three Dimensional Incompressible Flow

This is the most important kind fluid flow: the vast majority of physical phenomena take place in three dimensions and the velocities are small compared to the speed of sound.

### 4.1 The Clebsch Variables

In this case Euler equations can be expressed in a simpler form using a parametrization due to Clebsch [2]:

$$\omega = dq \wedge dp \tag{15}$$

It is clear that this is only possible because  $d\omega = 0$ . Locally, any one-form in  $R^3$  can be expressed in the form

$$a = d\lambda + qdp$$

so that any exact two form is locally of the form 15.

In terms of these variables, Euler equations become the statement that  $p, q$  are constant along streamlines:

$$\frac{\partial p}{\partial t} + v^i \partial_i p = 0, \quad \frac{\partial q}{\partial t} + v^i \partial_i q = 0$$

The velocity is determined in terms of  $p, q$  by

$$v^i = g^{ij} [\partial_j \lambda + q \partial_j p]$$

Here,  $\lambda$  is eliminated by the condition of incompressibility

$$\partial_i [\rho v^i] = 0 \iff \partial_i [\rho g^{ij} (\partial_j \lambda + q \partial_j p)] = 0.$$

#### 4.1.1 Canonical Relations

The Clebsch variables  $p, q$  are canonical conjugates of each other. That is, if we postulate canonical commutation relations

$$\{p(x), q(y)\} = \delta(x - y), \quad \{p(x), p(y)\} = 0 = \{q(x), q(y)\}$$

and the hamiltonian

$$H = \frac{1}{2} \int \rho v^i v^j g_{ij} dx$$

with  $v^i$  determined in terms of  $p, q$  as above, we get the Clebsch form of the Euler equations.

To see this note that the canonical commutation relations give

$$\{F, p(y)\} = -\frac{\delta F}{\delta q(y)}, \quad \{F, q(y)\} = \frac{\delta F}{\delta p(y)}$$

for any function of  $p, q$ . For example, if

$$F = \int f^{ij} \partial_i q \partial_j p dx$$

$$\{F, p(y)\} = -\partial_j f^{ij} \partial_i p, \quad \{F, q(y)\} = -\partial_j f^{ij} \partial_i q \quad (16)$$

if  $f^{ij}$  is independent of  $p, q$ . Now, the kinetic energy of the fluid can also be written as

$$H = \frac{1}{2} \int \rho v^i v^j g_{ij} dx = \frac{1}{4} \int f^{ij} \omega_{ij} dx = \frac{1}{2} \int f^{ij} \partial_i q \partial_j p$$

where  $\rho v^i = \partial_j f^{ij}$  and  $f^{ij} = -f^{ji}$  is the velocity potential. Therefore

$$\frac{\partial p}{\partial t} = \{H, p\} = -v^i \partial_i p, \quad \frac{\partial q}{\partial t} = \{H, q\} = -v^i \partial_i q$$

which are the Euler equations.

An extra factor of 2 appears (cancelling the  $\frac{1}{2}$  in the Hamiltonian) compared to 16 because  $f^{ij}$  itself depends linearly on  $v$  and hence on  $p, q$ .

#### 4.1.2 The Moment Map

Marsden and Weinstein [6] showed that the Clebsch parametrization has a natural geometric interpretation. On the space  $\mathcal{F}$  of real valued functions on a manifold with density, there is an inner product

$$\langle f, g \rangle = \int f g \rho dx.$$

This can be used to turn  $\mathcal{F} \oplus \mathcal{F}$  into a symplectic vector space: the conjugate pairs of functions  $p, q$  parametrize this phase space. Since volume preserving diffeomorphisms preserve the inner product above, they must act as canonical transformations on this phase space. The infinitesimal generator of these transformations is just vorticity. The Clebsch parametrization

$$\omega = dq \wedge dp$$

is analogous to the formula for angular momentum

$$\mathbf{L} = \mathbf{r} \times \mathbf{p}$$

in Euclidean space. The dot product is a rotation invariant inner product in  $R^3$ , which turns  $R^3 \oplus R^3$  into a phase space on which the rotations act as canonical transformations generated by  $\mathbf{L}$ .

#### A direction for further research

This point of view allows us to generalize the theory of a three dimensional incompressible fluid to the case where the underlying manifold is non-commutative sphere: a kind of ‘regularization’ of hydrodynamics where each point is replaced by a fuzzy object which is the average of many points. In addition to being more mathematically rigorous (the equations of motion are finite dimensional ODEs instead of PDEs) this might be a physically realistic description of the large scale behavior of fluids: the small scale fluctuations in velocity are averaged out to get an effective theory that is not no longer local.

## 4.2 The Lie Algebra of an Ideal Isentropic Fluid

The set of vector fields  $\mathcal{V}$  on a manifold is a Lie algebra under the commutator or Lie bracket,

$$[u, v]^i = u^k \partial_k v^i - v^k \partial_k u^i.$$

$\mathcal{V}$  acts on the space of scalars  $\mathcal{F}$  through the derivative

$$\mathcal{L}_u \phi = v^k \partial_k \phi.$$

The sum of the two vector spaces  $\mathcal{V} \oplus \mathcal{F} = \mathcal{G}$  is thus a Lie algebra as well:

$$\left[ (u, \phi), (\tilde{u}, \tilde{\phi}) \right] = \left( [u, \tilde{u}], \mathcal{L}_u \tilde{\phi} - \mathcal{L}_{\tilde{u}} \phi \right)$$

This is the semi-direct sum of the Lie algebra of vector fields and the abelian Lie algebra of scalars.

The dual of  $\mathcal{F}$  is the space of scalar densities  $\mathcal{F}^*$ :

$$i_\phi \rho = \int \phi \rho dx.$$

The dual of  $\mathcal{V}$  is the space of co-vector densities  $\mathcal{V}^*$ :

$$i_v j = \int v^i j_i dx.$$

Given functions  $F, G : \mathcal{V}^* \oplus \mathcal{F}^* \rightarrow \mathbb{R}$ , their Poisson brackets are given by

$$\{F, G\} = \int \rho \left( \frac{\delta F}{\delta j_k} \partial_k \frac{\delta G}{\delta \rho} - \frac{\delta G}{\delta j_k} \partial_k \frac{\delta F}{\delta \rho} \right) dx + \int j_i \left( \frac{\delta F}{\delta j_k} \partial_k \frac{\delta G}{\delta j_i} - \frac{\delta G}{\delta j_k} \partial_k \frac{\delta F}{\delta j_i} \right) dx$$

This can be written also as

$$\{F, G\} = \int \frac{\delta G}{\delta j_k} \left[ -\rho \partial_k \frac{\delta F}{\delta \rho} - j_i \partial_k \frac{\delta F}{\delta j_i} - \partial_i \left( j_k \frac{\delta F}{\delta j_i} \right) \right] dx - \int \frac{\delta G}{\delta \rho} \partial_k \left[ \rho \frac{\delta F}{\delta j_k} \right] dx.$$

Choosing  $G$  to be a linear function, we get a particularly convenient form of the Poisson bracket:

$$\begin{aligned} \{F, j_k\} &= -\rho \partial_k \frac{\delta F}{\delta \rho} - j_i \partial_k \frac{\delta F}{\delta j_i} - \partial_i \left( j_k \frac{\delta F}{\delta j_i} \right) \\ \{F, \rho\} &= -\partial_k \left[ \rho \frac{\delta F}{\delta j_k} \right] \end{aligned}$$

So far we have not needed any additional geometric structures such as a Riemannian metric: all the derivatives above make sense without the need of a connection.



Now we identify the variables  $j_i, \rho$  physically as momentum density and mass density respectively. The Hamiltonian is the sum of kinetic and potential (internal) energies of the fluid:

$$H = \int \frac{j_i j_j}{2\rho} g^{ij} dx + \int U(\rho) dx$$

where  $g_{ij}$  is a Riemannian metric on the underlying space.

On physical grounds we know also that momentum density is mass density times velocity:

$$j_i = g_{ik} \rho v^k$$

so that

$$\frac{\delta H}{\delta j_i} = v^i.$$

The equations of motion

$$\frac{\partial j_i}{\partial t} = \{H, j_i\}, \quad \frac{\partial \rho}{\partial t} = \{H, \rho\}$$

become

$$\begin{aligned} \frac{\partial j_i}{\partial t} &= -\rho \partial_k \left[ -\frac{1}{2} \frac{g^{ij} j_i j_j}{\rho^2} + \frac{\partial U}{\partial \rho} \right] - j_i \partial_k v^i - \partial_i (j_k v^i) \\ \frac{\partial \rho}{\partial t} &= -\partial_k (\rho v^k). \end{aligned}$$

We can simplify the first of these

$$\frac{\partial j_i}{\partial t} + \nabla_i (j_k v^i) = \frac{1}{2} \rho \partial_k [g_{ij} v^i v^j] - \rho v^i g_{ij} \nabla_k v^j - \rho \partial_k \left[ \frac{\partial U}{\partial \rho} \right]$$

where  $\nabla_i$  is the covariant derivative of the Riemann metric  $g_{ij}$ . The first two terms on the right hand side cancel each other. If we identify the enthalpy as

$$w = \frac{\partial U}{\partial \rho}$$

we get

$$\frac{\partial j_i}{\partial t} + \nabla_i (j_k v^i) + \rho \partial_k w = 0$$

or

$$\frac{\partial v^i}{\partial t} + v^k \nabla_k v^i + g^{ik} \partial_k w = 0$$

along with the conservation of mass:

$$\frac{\partial \rho}{\partial t} + \partial_k (\rho v^k) = 0.$$

These are exactly the Euler equations for an isentropic ideal fluid we obtained earlier, except that there we looked at the special case of the Euclidean metric. The Lie algebra  $\mathcal{V} \oplus \mathcal{F}$  as well as the Poisson algebra following from it are independent of the choice of metric. But the hamiltonian  $H$  depends on this choice. In this case, the hamiltonian is not a quadratic function, unlike in the case of the rigid body or the incompressible fluid. If the equation of state is a polytrope  $w = \rho^3$  it is possible to think of the hamiltonian as a cubic function. Perhaps this is worth deeper study.

## 5 Riemannian Geometry

We review here some facts about Riemannian manifolds, to rephrase some basic facts in a language convenient for our purposes. This is not meant as an introduction to Riemannian geometry. There are several excellent texts available, in particular the one by Chavel [9].

### 5.1 Covariant Derivative

A covariant derivative (connection)  $\nabla_u v$  of a vector field  $v$  on a manifold  $M$  along another vector field  $u$  should satisfy the conditions of linearity in  $u, v$  as well as

$$\nabla_u [fv] = f \nabla_u v + u(f)v, \quad \nabla_{fu} v = f \nabla_u v.$$

Thus it involves first derivatives of  $v$  and no derivative of  $u$ . Explicitly in co-ordinates

$$[\nabla_u v]^i = u^j \partial_j v^i + \Gamma_{jk}^i u^j v^k$$

for a set of connection coefficients  $\Gamma_{jk}^i$ . We will only be interested in connections without torsion:

$$\nabla_u v - \nabla_v u = [u, v].$$

The curvature is the tensor field defined by

$$R(u, v)w = \nabla_{[u, v]} w - \nabla_v \nabla_u w + \nabla_u \nabla_v w.$$

Given a Riemannian metric  $g$  on the manifold, there is a unique connection of zero torsion and which preserves the metric:

$$g(\nabla_u v, w) + g(v, \nabla_u w) = u(g(v, w)).$$

Explicitly in co-ordinates, this connection has coefficients

$$\Gamma_{jk}^i = \frac{1}{2}g^{il} \{ \partial_j g_{lk} + \partial_k g_{lj} - \partial_l g_{jk} \}$$

The curves that extremize the action

$$\int g(\dot{x}, \dot{x}) dt$$

are the geodesics; they satisfy the equation

$$\nabla_{\dot{x}} \dot{x} = 0$$

or

$$\ddot{x}^i + \Gamma_{jk}^i \dot{x}^j \dot{x}^k = 0.$$

It will be convenient to define the curvature bi-quadratic form

$$R(u, v) = g(R(u, v)v, u)$$

## 5.2 Geodesic Deviation and Curvature

The curvature form determines the rate of deviation of nearby geodesics from each other.

Consider a geodesic  $x^i(t)$ , expressed in a co-ordinate system centered at the initial point:  $x^i(0) = 0$ .

$$\frac{d^2 x^i}{dt^2} + \Gamma_{jk}^i \frac{dx^j}{dt} \frac{dx^k}{dt} = 0.$$

$$\frac{D\dot{x}}{dt} = 0.$$

Let the initial velocity be  $v \in T_0 M$ . An infinitesimally close geodesic to this one will satisfy

$$\frac{D^2 y^i}{dt^2} + R_{jkl}^i \frac{dx^j}{dt} \frac{dx^k}{dt} y^l = 0.$$

$$\frac{D^2 y}{dt^2} + R(\dot{x}, y)\dot{x} = 0.$$

The vector field  $y$  along the original geodesic connects the points at equal time on two nearby geodesics. It is called the Jacobi vector field. We can derive an equation for the length squared of the Jacobi field

$$\frac{1}{2} \frac{d^2 |y(t)|^2}{dt^2} = \left| \frac{Dy}{dt} \right|^2 - R(\dot{x}, y).$$

Suppose the initial conditions for the Jacobi equation are

$$y(0) = 0, \quad \frac{Dy^i}{dt}(0) = u, \quad \dot{x} = v.$$

That is, we consider two geodesics starting at the same point but with slightly different initial velocities.

Then

$$|y(t)|^2 = t^2|u|^2 - \frac{t^3}{3}R(u, v) + O(t^4).$$

The first term would have been the answer in Euclidean space. Thus, if  $R(u, v) < 0$ , the geodesic with tangent vector  $v$  is unstable with respect to an infinitesimal perturbation in the direction  $u$ .

### 5.3 Curvature as a Biquadratic

We saw that the curvature tensor on a Riemannian manifold describes the behavior of geodesics under small changes of the initial conditions. Therefore, it controls the stability properties of a physical system whose time evolution is along geodesics. It will be useful to calculate this tensor for left-invariant metrics on a Lie group to understand the stability of systems such as the rigid body or an ideal fluid. The formulas can get quite complicated, but a trick mentioned in Milnor's article allows us a simpler description. We will derive a simple formula that is quite useful in our applications by going a little beyond Milnor in this direction.

Recall that a co-variant symmetric tensor on a vector space is exactly the same thing as a quadratic form. A covariant symmetric tensor is a bilinear map  $Q : V \times V \rightarrow R$ , satisfying  $Q(u, v) = S(v, u)$ . A quadratic form is a function  $Q : V \rightarrow R$  that satisfies the scaling property

$$Q(\lambda u) = \lambda^2 Q(u).$$

Given a covariant symmetric tensor we can construct a quadratic by taking the special case when its entries are equal:

$$Q(u) = Q(u, u).$$

Conversely, we can get a symmetric tensor from a quadratic by 'polarization':

$$Q(u, v) = \frac{Q(u+v) - Q(u) - Q(v)}{2}.$$

These maps are inverses of each other.

In the same spirit, a tensor with the symmetries of the Riemannian curvature is fully determined by a bi-quadratic form on the tangent space. In the applications we have in mind, this is a much more convenient description, as we will be able to write explicit formulas and identify their positivity properties more easily.

Recall that a bi-quadratic is a function on a vector space  $T : V \times V \rightarrow R$  that satisfies

$$T(\lambda u, v) = \lambda^2 T(u, v), \quad T(u, \lambda v) = \lambda^2 T(u, v).$$

Now, the Riemann curvature tensor is a fourth rank tensor defined by

$$r(u, v, w, x) = g(\nabla_{[u, v]} w - \nabla_u \nabla_v w + \nabla_v \nabla_u w, x).$$

It defines a biquadratic on the tangent space by choosing  $w = u, v = x$  [8]

$$r(u, v) = g(\nabla_{[u, v]} u, v) - G(\nabla_u \nabla_v u, v) + G(\nabla_v \nabla_u u, v)$$

This function satisfies

$$r(\lambda u, v) = \lambda^2 r(u, v), \quad r(u, v) = r(v, u), \quad r(u, u) = 0. \quad (17)$$

The meaning of these conditions is that  $k(u, v) = \frac{r(u, v)}{g(u, u)g(v, v) - g(u, v)^2}$  depends on the subspace defined by  $u, v$ . To prove this, we just have to show that the condition

$$r(au + bv, cu + dv) = (ad - bc)^2 r(u, v) \quad (18)$$

is equivalent to the conditions (17). The denominator is the area of the parallelogram in the tangent space with sides  $u, v$ . The ratio  $k(u, v)$  is the sectional curvature of the plane spanned by  $u, v$ .

As Milnor points out, the conditions (17) imply in turn all the symmetry properties of a Riemann tensor. Also,  $k(u, v)$  determines fully the Riemann curvature tensor through a ‘polarization’ identity[9]:

$$r(u, v, w, x) = \frac{1}{6} \left[ \frac{\partial^2}{\partial s \partial t} \{r(u + sw, v + tx) - r(u + sx, v + tw)\} \right]_{s=t=0}.$$

The Ricci tensor is the trace of the Riemann tensor; equivalent to it is the quadratic form

$$r(u) = r(u, e_i, u, e_j) g^{ij}$$

where  $e_i$  is some basis in the tangent space and  $g^{ij}$  is the inverse of the matrix of inner products  $g_{ij} = g(e_i, e_j)$ .

From our current point of view it can be viewed as the average over all vectors

$$r(u) = \int r(u, v) d\mu_g(v).$$

The average is with respect to the Gaussian measure on the tangent space, with zero mean and covariance given by the metric  $g$ . Recall that in terms of components

$$\int v^i v^j d\mu_g(v) = g^{ij}$$

Also, the Ricci scalar is the further average over  $u$ :

$$r = \int r(u) d\mu_g(u).$$

This point of view has the advantage that it extends to infinite dimensions. Also it is more natural in applications involving random forces.

## 6 Geometry of Left-Invariant Metrics

Let  $\mathfrak{G}$  be a Lie group. Its Lie algebra  $\mathcal{G}$  can be thought of either as the tangent space at the identity or as the space of left invariant vector fields on  $\mathfrak{G}$ . An inner product  $G$  on  $\mathcal{G}$  is thus equivalent to a left-invariant Riemannian metric on  $\mathfrak{G}$ . We will study the geodesics and curvature of this metric. Using homogeneity, all computations can be reduced to the Lie algebra. The basic reference is the article by Milnor [8], especially Section 5. We will go beyond Milnor in deriving explicit formulas in a form useful for applications to mechanics.

### 6.1 Covariant Derivative

The covariant derivative (Levi-Civita connection) is determined by the derivative of a left-invariant vector field by another. We will denote the covariant derivative on a group manifold by  $D$  to distinguish it from the covariant derivative on a general Riemannian manifold, which we denote by  $\nabla$ . This will be helpful when we talk of the diffeomorphism group of a Riemannian manifold: the derivative on the group of diffeomorphisms and that on the underlying manifold are closely related, but not identical notions.

The conditions of zero torsion and preserving the metric become

$$\begin{aligned} D_u v - D_v u &= [u, v], \\ G(D_u v, w) + G(v, D_u w) &= 0; \end{aligned}$$

The zero torsion condition can also be written as

$$G(D_u v, w) - G(D_v u, w) = G([u, v], w).$$

Its cyclic permutations give

$$G(D_v w, u) - G(D_w v, u) = G([v, w], u)$$

$$G(D_w u, v) - G(D_u w, v) = G([w, u], v)$$

Adding the first and third and subtracting the second, and using the invariance of the metric

$$G(D_u v, w) = \frac{1}{2} \{G([u, v], w) - G([v, w], u) + G([w, u], v)\}$$

Define the linear operator  $\tilde{u} : \mathcal{G} \rightarrow \mathcal{G}$  by

$$G(\tilde{u}v, w) = G([u, v], w) + G(v, [u, w]). \quad (19)$$

If the metric  $G$  were invariant under the Lie algebra,  $\tilde{u}$  would vanish for all  $u$ . Thus it measures the deformation of the metric. Now,

$$G(\tilde{u}v, w) + G(\tilde{v}u, w) = G([u, w], v) + G([v, w], u). \quad (20)$$

Thus

$$D_u v = \frac{1}{2} \{[u, v] - \tilde{u}v - \tilde{v}u\}.$$

## 6.2 Geodesics on a Group Manifold

Given a curve  $\gamma : [a, b] \rightarrow \mathfrak{G}$ , its tangent vector at each point can be thought of as an element of the Lie algebra:

$$\frac{d\gamma}{dt} = \gamma v.$$

Thus the tangent vectors give a curve in the Lie algebra  $v : [a, b] \rightarrow \mathcal{G}$ .

We can define the action of the curve as

$$S(\gamma) = \frac{1}{2} \int_a^b G(v, v) dt.$$

(Some geometers call this quantity the energy; physicists would call it the action.)

A geodesic is a curve at which the action is stationary with respect to small variations.

To compute this variation, let us consider a one parameter family of curves; that is a map

$\phi : [0, \epsilon] \times [a, b] \rightarrow \mathfrak{G}$ , for some positive number  $\epsilon$ . We require that the initial and final points are left unchanged  $\phi(s, a)$  and  $\phi(s, b)$  are independent of  $s$ .

Define

$$\frac{\partial \phi}{\partial t} = \phi v, \quad \frac{\partial \phi}{\partial s} = \phi u.$$

So  $u(s, a) = u(s, b) = 0$ .

We get the integrability condition

$$\partial_s v = \partial_t u + [v, u].$$

Regarding  $\phi(\cdot, s)$  for each value of  $s$  as a curve, the action becomes a function of  $s$ . Its derivative is

$$\partial_s S(\gamma_s) = \int_a^b G(\partial_s v, v) dt = \int_a^b G(\partial_t u + [v, u], v) dt$$

Using the linear operator  $\tilde{v} : \mathcal{G} \rightarrow \mathcal{G}$  defined earlier

$$G([v, u], v) = G(u, \tilde{v}v).$$

By integration by parts

$$\partial_s S(\gamma_s) = - \int_a^b G(u, \partial_t v - \tilde{v}v) dt.$$

Thus the geodesic equation on the group becomes the ODE on the Lie algebra

$$\partial_t v + D_v v = 0. \quad (21)$$

### 6.3 Curvature of a Left-Invariant Metric

We will now calculate explicitly the curvature bi-quadratic for a Metric Lie Algebra; i.e., for a Lie algebra  $(\mathcal{G}, G)$  with an inner product on it.

$$G(D_v D_u u, v) = -G(D_u u, D_v v) = -G(\tilde{u}u, \tilde{v}v).$$

$$-G(D_u D_v u, v) = G(D_v u, D_u v) = -\frac{1}{4}|[u, v]|^2 + \frac{1}{4}|\tilde{u}v + \tilde{v}u|^2.$$

$$G(D_{[u, v]} u, v) = \frac{1}{2}G([u, v], u, v) - \frac{1}{2}|[u, v]|^2 + \frac{1}{2}G([v, [u, v]], u)$$

Thus

$$R(u, v) = -\frac{3}{4}|[u, v]|^2 + \frac{1}{2}\{G([u, v], u, v) + G([v, [u, v]], u)\} + \frac{1}{4}|\tilde{u}v + \tilde{v}u|^2 - G(\tilde{u}u, \tilde{v}v)$$

The middle term can be further simplified using

$$G([u, w], v) = G(\tilde{u}w, v) - G(w, [u, v])$$

and  $G(\tilde{u}w, v) = G(\tilde{u}v, w)$ . We get

$$\frac{1}{2}\{G([u, v], u, v) + G([v, [u, v]], u)\} = |[u, v]|^2 + \frac{1}{2}G([u, v], \tilde{v}u - \tilde{u}v).$$

Thus

$$R(u, v) = \frac{1}{4}|[u, v]|^2 + \frac{1}{2}G([u, v], \tilde{v}u - \tilde{u}v) + \frac{1}{4}|\tilde{u}v + \tilde{v}u|^2 - G(\tilde{u}u, \tilde{v}v)$$



If the metric is bi-invariant, only the first term survives: the curvature of a bi-invariant metric is postive. We can ‘complete the square’ on the first two terms to get

$$R(u, v) = \frac{1}{4} |[u, v] + \tilde{v}u - \tilde{u}v|^2 + G(\tilde{u}v, \tilde{v}u) - G(\tilde{u}u, \tilde{v}v) \quad (22)$$

This simple formula appears to be a new result.

We already see something important: the sectional curvature in any plane that contains a Killing vector is positive:

$$\tilde{u} = 0 \implies R(u, v) = \frac{1}{4} |[u, v] + \tilde{v}u|^2.$$

This result is known but is hard to see with the formulas known in the literature.

## 6.4 Example: The Two Dimensional Lie Algebra

The only non-abelian Lie algebra of dimension two has commutation relations in terms of basis vectors

$$[e_0, e_1] = e_1$$

or in terms of components

$$[u, v] = (0, u_0v_1 - v_0u_1).$$

The only metric is (up to a change of basis that does not change the above commutation relations)

$$G(u, v) = u_0v_0 + u_1v_1.$$

The corresponding group manifold is the half plane,

$$H = \{(x_0, x_1) | x_0 > 0\}$$

with the multiplication law

$$(x_0, x_1)(x'_0, x'_1) = (x_0x'_0, x_0x'_1 + x_1).$$

The corresponding left-invariant metric is the Poincare metric. It is well known that this metric has constant negative curvature. Hence it can serve as a simple model of an unstable dynamical system.

We get

$$G(\tilde{v}v, w) = G(v, [v, w]) = v_1(v_0w_1 - w_0v_1)$$

so that

$$\tilde{v}v = (-v_1^2, v_0v_1).$$

The geodesic equation in the Lie algebra becomes

$$\frac{dv_0}{dt} = -v_1^2, \quad \frac{dv_1}{dt} = v_0 v_1 \quad (23)$$

The solution is (scaling  $t$  so that  $|v| = 1$ )

$$v_0 = -\tanh t, \quad v_1 = \frac{1}{\cosh t}.$$

To get the curve in the group we must solve

$$\frac{dg}{dt} = gv$$

where  $gv$  denotes the action of the group on its Lie algebra. Viewing  $v$  as an element infinitesimally close to the identity

$$(x_0, x_1)v = (x_0 v_0, x_0 v_1)$$

Thus

$$\frac{dx_0}{dt} = x_0 v_0, \quad \frac{dx_1}{dt} = x_0 v_1. \quad (24)$$

The Riemannian metric is thus the Poincare' metric

$$dt^2 = \frac{dx_0^2 + dx_1^2}{x_0^2}.$$

Thus geodesics are

$$\left(\frac{A}{\cosh(t - t_0)}, B + A \tanh[t - t_0]\right).$$

(It is convenient to choose the origin of time as the point where  $v_0(t)$  vanishes instead of the starting time.) These are the well-known semicircles

$$x_0^2(t) + (x_1(t) - B)^2 = A^2.$$

#### 6.4.1 Conserved Quantities of Geodesic Motion

The Killing vectors of the Poincare' metric

$$f_0 = x_0 \partial_0 + x_1 \partial_1, \quad f_1 = \partial_1, \quad f_{-1} = (x_1^2 - x_0^2) \partial_1 + 2x_0 x_1 \partial_0$$

form the  $\mathcal{SL}(2, R)$  Lie algebra:

$$[f_0, f_1] = f_1, \quad [f_0, f_{-1}] = f_{-1}, \quad [f_1, f_{-1}] = 2f_0.$$

The corresponding conserved quantities of geodesic motion are

$$F_0 = x_0 p_0 + x_1 p_1, \quad F_1 = p_1, \quad F_{-1} = (x_1^2 - x_0^2) p_1 + 2x_0 x_1 p_0$$

where  $p_a$  are the canonical conjugates of  $x^a$  :

$$\{p_a, x_b\} = \delta_{ab}.$$

Since the geodesic equations have as hamiltonian

$$H = \frac{1}{2}g^{ab}(x)p_ap_b$$

we get  $\frac{dx_a}{dt} = g^{ab}p_b$ . Expressing in terms of the variables  $v_a$  introduced earlier

$$p_a = \frac{v_a}{x_0}.$$

Thus we see that the conserved quantities of the equations (23,24) above are

$$F_0 = v_0 + \frac{x_1}{x_0}v_1, \quad F_1 = \frac{v_1}{x_0}, \quad F_{-1} = \frac{x_1^2 - x_0^2}{x_0}v_1 + 2x_1v_0.$$

Of these,  $F_0, F_1$  generate the left translations of the half-plane thought of as a group. Now, note that  $F_0^2 - F_1F_{-1} = v_0^2 + v_1^2$  which is an ‘obvious’ conserved quantity. Putting in the formula for the  $L$ ’s we get the semi-circle for the shape of the geodesic.

### Curvature Form

It is straightforward to calculate the curvature biquadratic for this case:

$$R(u, v) = -[u_0v_1 - v_0u_1]^2$$

which simply says that the sectional curvature is:

$$K(u, v) = -1.$$

We use this simple case to check the numerical factors in our formula for curvature.

## 6.5 Geodesics on $SO(3)$

Consider  $R^3$  as a Lie algebra with the cross-product as the Lie bracket:

$$[u, v] = (u_2v_3 - u_3v_2, u_3v_1 - u_1v_3, v_2 - u_2v_1).$$

A corresponding Lie group is  $SO(3)$ . Any inner product in  $R^3$  can be brought to the diagonal form by a rotation without changing the Lie bracket:

$$G(u, v) = G_1u_1v_1 + G_2u_2v_2 + G_3u_3v_3$$

Thus

$$G(\tilde{u}v, w) = G_1 \{ [u_2v_3 - u_3v_2] w_1 + [u_2w_3 - u_3w_2] v_1 \}$$

$$+G_2 \{ [u_3 v_1 - u_1 v_3] w_2 + [u_3 w_1 - u_1 w_3] v_2 \}$$

$$+G_3 \{ [u_1 v_2 - u_2 v_1] w_3 + [u_1 w_2 - u_2 w_1] v_3 \}$$

so that

$$[\tilde{u}v]_1 = \frac{G_1 - G_3}{G_1} u_2 v_3 + \frac{G_2 - G_1}{G_1} u_3 v_2, \dots$$

The dots denote three more relations obtained by cyclic permutations. In particular,

$$[\tilde{v}v]_1 = \frac{G_2 - G_3}{G_1} v_2 v_3$$

The geodesic equations become

$$\frac{dv_1}{dt} + \frac{G_3 - G_2}{G_1} v_2 v_3 = 0, \dots$$

Calculating as above gives the formula for curvature

$$R(u, v) = \frac{(G_2 - G_3)^2 + 2G_1(G_2 + G_3) - 3G_1^2}{4G_1} (u_2 v_3 - v_2 u_3)^2 + \dots$$

In particular, if we choose  $u = (0, \frac{1}{\sqrt{G_2}}, 0)$  to be a unit vector in the second principal direction and  $v$  to be a unit vector in the third direction the sectional curvature of the 23-plane is

$$K_{23} = \frac{(G_2 - G_3)^2 + 2G_1(G_2 + G_3) - 3G_1^2}{4G_1 G_2 G_3}.$$

The others are given by cyclic permutations.

### 6.5.1 Stability of the Rigid Body

Even for an anisotropic rigid body with  $G_1 < G_2 < G_3$ , the rotations around the principal axes are time-independent:  $(v_1, 0, 0)$  for example is a solution of the Euler equations [1]. Under small perturbations, the rotations around the principal axes with the largest and smallest moment of inertia are stable. But a rotation around the principal axis with the middle value of moment of inertia is unstable: small perturbations grow exponentially.

The geometric picture allows us to generalize this analysis to time-dependent solutions. The geodesic deviation equation shows that a small perturbation along  $u$  to a geodesic with tangent vector  $v$  will grow exponentially if  $R(u, v) < 0$ . Let us see under what conditions this is possible. A change of variables from the principal moments of inertia to the principal curvatures of the body will help us.

Recall that the principal moments of inertia are

$$G_1 = M_2 + M_3, \dots$$

where  $M_1, M_2, M_3$  are the (always positive) eigenvalues of the moment matrix  $M = \int \rho x \otimes x dx$ .

Define their reciprocals

$$\mu_1 = \frac{1}{M_1} = \left[ \int \rho(x) x_1^2 dx \right]^{-1}, \dots$$

In terms of these,

$$\begin{aligned} (G_2 - G_3)^2 + 2G_1(G_2 + G_3) - 3G_1^2 &= 4[M_1 M_2 + M_1 M_3 - M_2 M_3] \cdot \\ &= \frac{4}{\mu_1 \mu_2 \mu_3} [\mu_2 + \mu_3 - \mu_1] \end{aligned}$$

$$R(u, v) = \frac{\mu_2 + \mu_3 - \mu_1}{\mu_1(\mu_2 + \mu_3)} [u_2 v_3 - u_3 v_2]^2 + \dots$$

Thus, if the three principals  $\mu_1, \mu_2, \mu_3$  satisfy the triangle inequalities,

$$\mu_1 + \mu_2 > \mu_3, \quad \mu_2 + \mu_3 > \mu_1, \quad \mu_3 + \mu_1 > \mu_2$$

the curvature  $R(u, v)$  will be positive for all pairs: a rigid body that is not ‘too anisotropic’ is geodesically stable. But these inequalities can be violated for a ‘flat enough’ shape. Let us look at an example.

### Stability of a rotating cylinder

If the body has an axis of symmetry (say the third axis)  $\mu_1 = \mu_2$  and

$$\begin{aligned} R(u, v) &= \frac{\mu_3}{\mu_1(\mu_1 + \mu_3)} \{ [u_2 v_3 - u_3 v_2]^2 + [u_1 v_3 - u_3 v_1]^2 \} \\ &\quad + \frac{2\mu_1 - \mu_3}{2\mu_1 \mu_3} [u_1 v_2 - u_2 v_1]^2 \end{aligned}$$

So an instability arises if

$$\mu_3 > 2\mu_1.$$

Consider a rotating cylinder of uniform density. Its height is  $h$  and its radius is  $r$ . It is elementary that

$$\begin{aligned} M_1 = M_2 &= \frac{r^2}{4}, \quad M_3 = \frac{h^2}{12}. \\ \mu_1 = \mu_2 &= \frac{4}{r^2}, \quad \mu_3 = \frac{12}{h^2}. \end{aligned}$$

Thus a cylinder with too small a height

$$h \leq \sqrt{\frac{3}{2}}r$$

will be unstable. If it is rotating around one of its diameters, it is unstable with respect to a change of axis towards one of the other diameters. An example of such motion is a coin toss.

## 7 The Geometry of Diffeomorphisms

It is a remarkable fact that the set of Diffeomorphisms of a Riemannian manifold is itself a Riemannian manifold. This "higher Riemannian geometry" is the proper language of many interesting physical systems such as ideal fluids. Such repetitions of structures at a higher level happen quite often in mathematics: the set of Riemannian metrics on a manifold is itself a Riemannian manifold; the set of complex structures is a complex manifold; the set of Kähler structures is itself a Kähler manifold and so on.

### 7.1 The Circle

The simplest manifold is the circle. So the first example of a Diffeomorphism group must be  $Diff(S^1)$ . The standard metric on the circle leads to the  $L^2$ -metric on this group.

$$G(u, v) = \int_0^{2\pi} u(x)v(x)dx.$$

The deformation tensor of a vector field on the circle is easily found from

$$G(\tilde{u}v, w) = G(uv' - vu', w) + G(v, uw' - wu') = \int \{uv'w - vu'w + vwu' - vwu'\} dx.$$

to be

$$\tilde{u} = -3u'.$$

Thus the covariant derivative on the Lie algebra is

$$D_u v = \frac{1}{2} \{[u, v] - \tilde{u}v - \tilde{v}u\} = u'v + 2uv'.$$

The geodesic equation is the inviscid Burger's equation

$$\frac{\partial v}{\partial t} + 3v \frac{\partial v}{\partial x} = 0.$$

The curvature form

$$R(u, v) = \frac{1}{4} |[u, v] + \tilde{v}u - \tilde{u}v|^2 + G(\tilde{u}v, \tilde{v}u) - G(\tilde{u}u, \tilde{v}v)$$

reduces to

$$R(u, v) = |[u, v]|^2.$$

Thus the  $L^2$ -metric on  $Diff(S^1)$  has positive curvature.

### A Technical Remark

This contradicts the statement in [4] and in several other places in the mathematics literature that the  $L^2$ -metric on the group of diffeomorphisms of  $M$  has zero curvature if  $(M, g)$  is itself flat. In particular, it is not possible to calculate the curvature of the subgroup of incompressible diffeomorphisms by using the Gauss-Codazzi formula as is often claimed. This is despite the fact the final answer has an uncanny resemblance to this formula. The confusion arises because there is a connection on  $Diff(M)$  whose curvature form is the average of the curvature form of  $M$ . But it does not preserve the metric on the group manifold:

$$\int g(\nabla_u v, w) dv_g + \int g(v, \nabla_v w) dv_g \neq 0$$

where  $dv_g$  is the Riemannian volume element on  $M$ . We take a more direct route in the next section to avoid this pitfall.

## 7.2 Incompressible Diffeomorphisms

Given a scalar density on a manifold, the set of diffeomorphisms that preserve it form a subgroup:

$$\mathfrak{D}_\rho = \{\phi \in \mathfrak{D} | \rho(\phi(x)) = \det \partial \phi(x) \rho(x)\}.$$

Its Lie algebra is the set of vector fields of zero divergence. Recall that the divergence of a vector field is defined with respect to a density as

$$\operatorname{div} u = \frac{1}{\rho} \partial_i [\rho u^i].$$

From the identity

$$\operatorname{div}[u, v] = u[\operatorname{div} v] - v[\operatorname{div} u]$$

it follows that incompressible (divergenceless) vector fields form a sub-Lie algebra, which we will call  $\mathcal{G}$ .

Given a Riemannian metric  $g$  on  $M$  there is an inner product on the space of vector fields on  $M$ :

$$G(u, v) = \int g(u, v) \rho.$$

We will call this the  $L^2$ -inner product even though we are not completing the space of vector fields with respect to the induced norm. Define now the *deformation tensor*  $\tilde{u}$  of a vector field:

$$\begin{aligned} G(\tilde{u}v, w) &= \int \{g([u, v], w) + g(v, [u, w])\} \rho \\ &= \int \{g(\nabla_u v - \nabla_v u, w) + g(v, \nabla_u w - \nabla_w u)\} \rho \end{aligned}$$

where  $\nabla_u$  is the Riemannian covariant derivative on the manifold  $(M, g)$ . Thus

$$G(\tilde{u}v, w) = \int \{\nabla_u [g(v, w)]\} \rho - \int v^i w^j [\nabla_i u_j + \nabla_j u_i] \rho$$

After an integration by parts the first term is zero, when  $u$  has zero divergence. It follows that

$$[\tilde{u}v]_j = -[\nabla_i u_j + \nabla_j u_i] v^j + \nabla_i \phi(u, v)$$

where  $\phi(u, v)$  is to be chosen such that the lhs has zero divergence:

$$\nabla^i \{-[\nabla_i u_j + \nabla_j u_i] v^j + \nabla_i \phi(u, v)\} = 0$$

Thus, absorbing  $\nabla(g(u, v))$  into the gradient term,

$$[\tilde{u}v + \tilde{v}u] = -[\nabla_v u + \nabla_u v] + \nabla_i p(u, v)$$

$$[D_u v] = \frac{1}{2} \{[u, v] + \nabla_v u + \nabla_u v\} + \nabla p(u, v)$$

But

$$[u, v] = \nabla_u v - \nabla_v u$$

so that

$$[D_u v] = \nabla_v u + \nabla p(u, v), \quad \nabla^2 p(u, v) + \operatorname{div} \nabla_v u = 0.$$

In particular

$$D_v v = \nabla_v v + \nabla p$$

which  $p$  chosen such that the divergence is zero. So the geodesic equation on the group of volume preserving diffeomorphisms is the Euler equation, as promised.

The  $L^2$ -inner product allows us to split the space of vector fields into the subspace of divergence-free vector fields (which is also a subalgebra) and another subspace of gradients:



$$u = u^T + \nabla \phi(u), \quad \operatorname{div} u^T = 0, \quad \nabla^2 \phi(u) = \operatorname{div} u.$$

This is an orthogonal decomposition. The Riemannian covariant derivative in the group of volume preserving diffeomorphisms is just the transverse projection of the covariant derivative on  $(M, g)$ :

$$D_u v = [\nabla_u v]^T.$$

Thus we can split the  $L^2$ -inner product on vector fields into transverse and longitudinal pieces

$$G(u, v) = T(u, v) + S(u, v)$$

where

$$T(u, v) = G(u^T, v^T), \quad S(u, v) = G(\nabla \phi(u), \nabla \phi(v)).$$

In other words, even if  $w$  is not of zero divergence

$$G(D_u v, w) = T(\nabla_u v, w)$$

### 7.3 Curvature of the Diffeomorphism Group

$$R(u, v) = G(D_{[u, v]} u, v) - G(D_u D_v u, v) + G(D_v D_u u, v)$$

$$= G(D_{[u, v]} u, v) + G(D_v u, D_u v) - G(D_u u, D_v v)$$

$$= G(\nabla_{[u, v]} u, v) + T(\nabla_v u, \nabla_u v) - T(\nabla_u u, \nabla_v v)$$

where we use the fact that  $v$  has zero divergence.

Now,

$$\nabla_{[u, v]} u =$$

$$= r(u, v)u + \nabla_u \nabla_v u - \nabla_v \nabla_u u$$

$$R(u, v) = \int r(u, v) \rho + G(\nabla_u \nabla_v u - \nabla_v \nabla_u u, v) + T(\nabla_v u, \nabla_u v) - T(\nabla_u u, \nabla_v v)$$

Set  $w = \nabla_v u$ , which may not have zero divergence, even though  $u$  and  $v$  are of zero divergence. Then,

$$\begin{aligned} G(\nabla_u w, v) &= \int g(\nabla_u w, v) \rho \\ &= \int \nabla_u [g(w, v)] \rho - \int g(w, \nabla_u v) \rho \end{aligned}$$

$$= -G(w, \nabla_u v) = -S(w, \nabla_u v) - T(w, \nabla_u v)$$

And similarly for  $G(\nabla_v \nabla_u u, v)$ . Thus

$$R(u, v) = \bar{r}(u, v) + S(\nabla_u u, \nabla_v v) - S(\nabla_v u, \nabla_u v)$$

where

$$\bar{r}(u, v) = \int r(u, v) \rho$$

is the curvature form of  $(M, g)$  averaged by the density. It is not difficult now to work out explicit answers for a flat torus and recover Arnold's original results. We find the present form more useful as well as more general.

## 8 Conclusion

Arnold's geometric explanation of the instabilities of an ideal fluid raises an important physical question. Do the Euler equations describe the time evolution of real fluids? Any theoretical description of a physical system is an idealization in which small forces are ignored. This is usually justified because small forces have small effects.

Near an unstable point this breaks down. Just think of a pendulum balanced precariously on its head, at the unstable equilibrium point. A small force in either direction can topple it in one direction or the other. The final outcome is random, since no one can predict with certainty such tiny perturbations.

What if a system is dynamically unstable at every point in its phase space? This is the case with an ideal fluid. In this case no where is it justified to ignore the unpredictable small forces. Instead of predicting these forces, we must model them as a stochastic process; for example, as Gaussian white noise. The time evolution of the system is then a stochastic differential equation.

What is the evolution of an unstable dynamical system under infinitesimally small white noise random forces? In a later paper [10] we will show that this can be reduced to a *deterministic* dynamical system, but with *twice as many* degrees of freedom. The Euler equations will then get replaced with equations for geodesics on the tangent bundle of the diffeomorphism group.

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